Can we extend the method of images in electrostatics to non-trivial cases?

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What do we like to do here:

The method of images is a powerful technique in electrostatics for computing the electric potential in special systems involving grounded conductors and charges. This topic is typically studied in upper-level undergraduate electromagnetism (such as in our PHY 371). However, within this category of electrostatics problems, only the simplest examples are worked out in electromagnetism textbooks, so it would be interesting to test our courage to extend the method beyond the simplest cases. We take up this task in this poster, first review the basics and a simple example, and then we take our first step in the direction of extension (our minimal extension). In the next set of posters, we will consider more non-trivial extensions.

A side note:

In undergraduate physics courses, we typically deal with solvable systems. Needless to say, some of these systems can get quite complicated and nontrivial to analyze, but they are, in principle, solvable (or at least their framework is known). However, in the forefront of the contemporary theoretical physics, there are systems that, within the current state of knowledge, are not exactly solvable and/or their frameworks are not yet fully understood. Nevertheless, over the past several decades, there have been major attempts in tackling unsolved problems by studying their theories in useful limits that result in connections to other theories. The word "duality" is typically used when two theories converge in certain limits or under special conditions. Interplays between such different theories have provided new insights. For example, connections between quantum theory of gravity (expressed by string theory) and the quantum field theories (such as those used for description of elementary particles), have resulted in many interesting investigations.

When we study the method of images in electrostatics, we find that two completely different systems have exactly the same physical properties (within a certain region of space). I like to think of these two systems as being the "dual" of each other, and this, in my mind, somehow resembles the notion of duality just mentioned, but within a classical context. The difference here is that the duality is between two systems studied within the same theory, but still is a type of duality.

Let's first refresh our memory a bit:

If we bring a charged object (like a charged rod made of, for example, hard rubber or glass which are the type of rods we use in lab) close to a neutral and isolated conducting sphere, the charges on the sphere separate from each other and we have something like shown in Fig. 1. This phenomenon is known as induction.

Fig. 1: Induction in an isolated conducting sphere.

Now, if we ground the conducting sphere some of the charges on the sphere (on the opposite side of the rod) get neutralized by the ground and we end up with something like what is shown in Fig. 2.

Fig. 2: Induction in a grounded conducting sphere.

Since the sphere is connected to the ground the potential on the sphere is at the same potential of the Earth which is conveniently taken to be zero.

Is it easy to perform this experiment?

Despite being a phenomenon that is easily understandable, its experimental observation is tricky. The reason is that many factors interfere with this experiment, such as many sources of static charge that are present in the lab or on the clothes of the experimenter. Do we remember these issues in PHY 202 L? Well, this is what we did:

The typical equipment for measuring static charge is consist of a Faraday cage connected to an electrometer as shown in Fig. 3. We perform this experiment in our PHY 102L and PHY 202L. Using a special tool for collecting charge, we collect charge samples from different parts of the aluminum sphere and enter them into the inner cage and read the numbers displayed by the electrometer which tell us the sign of the samples as well as, qualitatively, the amount of the charge in each sample.

Fig. 3: Faraday cage connected to an electrometer used in measurement of static charge.

However, practically, these readings are not always consistent and stable. Therefore, while the idea of this experiment is so simple, its performance is not always that easy due to different sources of charge contamination and interference with surrounding.

How about theoretical calculation?

Can we theoretically calculate the charge distribution on the aluminum sphere? As we will see next, the theoretical calculation is also not that trivial either, and depending on the geometry of the charged object that we bring near the grounded sphere, the solution can be quite elaborate.

To develop a method for theoretically modeling the charge distribution on the aluminum sphere of Fig. 1 and 2, as our first step, we can start from an oversimplified situation in which we only have one point charge in front of the sphere. Once we understand this oversimplified case, we can use its formulation as a basis for more involved charged distributions, such as, ultimately, a distribution that closely resembles the charged rod of Fig. 1 & 2.

Fortunately, the point charge case is treated in all upper division electromagnetism textbooks, such as the "Introduction to Electrodynamics," by D.J. Griffiths. The set up and notation is shown in Fig. 4.

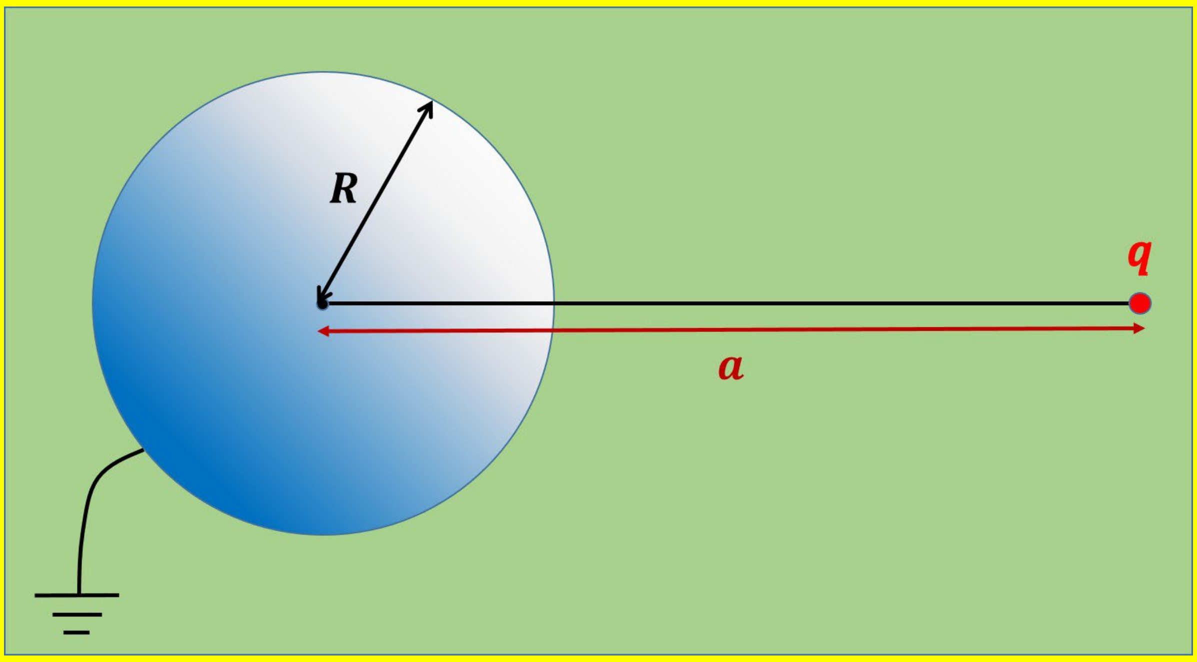
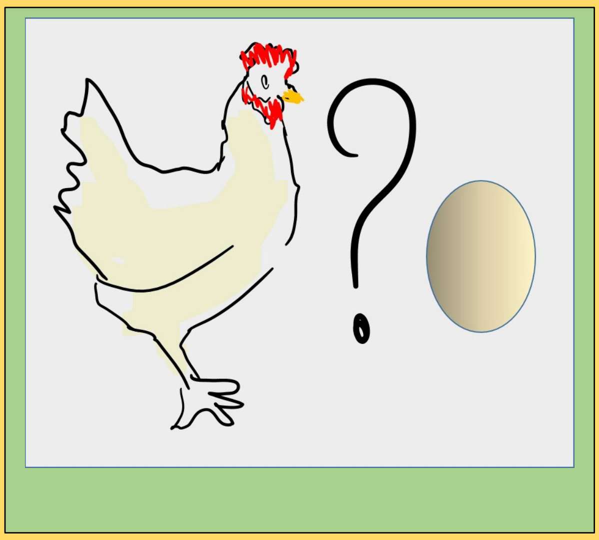


Fig. 4: A point charge in front of a grounded conducting sphere

Now, is it easy to find the electric potential everywhere in the system of Fig. 4 (obviously we mean outside the sphere)? At first, one may think that the answer is yes because there is only one point charge in this system. But, this is NOT correct, and here is why:

Note that, due to induction, a charge distribution accumulates on the surface of the conducting grounded sphere, but, a priori, we don't know this distribution! In other words, to find the electric potential, we need this charge distribution, and to find the charge distribution we need to know the electric potential, therefore, we seem to be dealing with a classic chicken and egg problem!



What comes to our rescue out of this apparent gridlock is the mathematical foundation of electrostatics. For this system we need to solve the Poisson partial differential equation:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \qquad (1)$$

This PDE rests on several Uniqueness Theorems that specify situations in which the solution of this PDE is unique. For example, one of these theorems states that if the charge distribution as well as the electric potential is specified in a system (which is the case here), then the solution to this PDE is uniquely determined.

What is so special about this uniqueness of solution? It authorizes us to come up with a solution by any means, including guessing a solution, because if we can find one solution, then it has to be the ONLY one!

A method that helps us figure out the solution is known as the method of images. This method kind of resembles the optical image of objects in flat or curved mirrors. Effectively, this method shows that the system consist of a point charge in front of a grounded conductor can be traded with another system in which the effect of the conductor is described by an "image charge." Let us examine this for our system (learn this technique in PHY 371).

According to this method, we find that, amazingly, the electric potential for every point outside (or on) the aluminum sphere is the same as the electric potential in the very simple two point charge system (the original charge q and its image charge q') depicted in Fig. 5 for distances

$$r \geq R!!!$$

I like to call these two completely different systems "Dual Systems" because they have exactly the same physics outside the sphere!

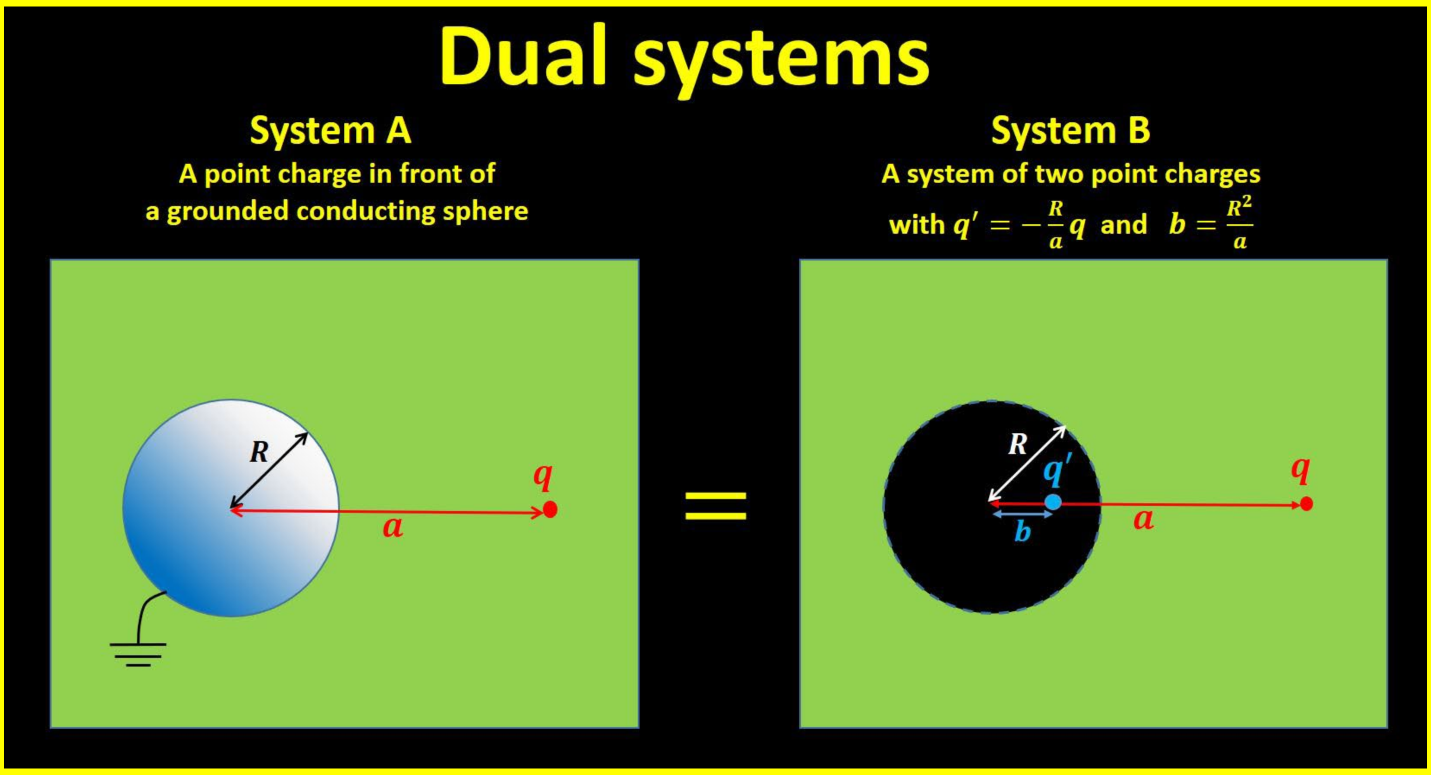


Fig. 5: Dual electrostatics systems

As given in the figure, the exact characteristics of the dual system of two point charges (the original charge + its image charge) are:

$$q' = -\frac{R}{a}q \tag{2}$$

$$b = \frac{R^2}{a} \tag{3}$$

Now that we know the magical solution, we can calculate anything we like (such as the charge distribution on the surface of the conductor that we were initially so puzzled about), using the exact solution given by the dual system. The electric potential anywhere in either system is determined by:

$$V(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|r-r_q|} + \frac{q'}{|r-r_{q'}|}\right) \tag{4}$$

so simple, AND, exact!

Then, using this exact solution, the "mystery" charge-density can be calculated as:

$$\sigma = -\epsilon_0 \frac{\partial V(r)}{\partial r}|_{r=R} \tag{5}$$

$$\sigma = \frac{q}{4\pi R}\left(R^2 - a^2\right)\left(R^2 + a^2 - 2Ra\cos\theta\right)^{-\frac{3}{2}} \tag{6}$$

This is already an amazing solution, however, this formulation is only for a single point charge in front of a grounded conducting sphere. It is interesting to try to see if we can extend this method to our original case of a charged rod in front of a grounded conducting sphere!

How do we tackle the rod case?

Equipped with the solution for the simplest case of a point charge, we can now think about non-trivial extensions (that go beyond the level of textbooks).

In this poster we discuss the simplest case of an extended charged object, i.e. a uniformly charged rod along the radial direction shown in Fig. 6. We can connect this case to the previous discussion of a point charge case, by considering an infinitesimal portion of the charge dq on this rod, which, according to previous case, has an image dq' inside the sphere as shown. The magnitude and location of the image can be worked out from Eqs. (2) and (3). Clearly, this immediately suggests that the image of this rod is a rod inside the sphere which turns out to be true (but, the general case is not so easy, and it can be shown that the shape of the image, in general, has nothing to do with the shape of the charged object -- see next posters!). Our parametrization is defined in Fig. 7.

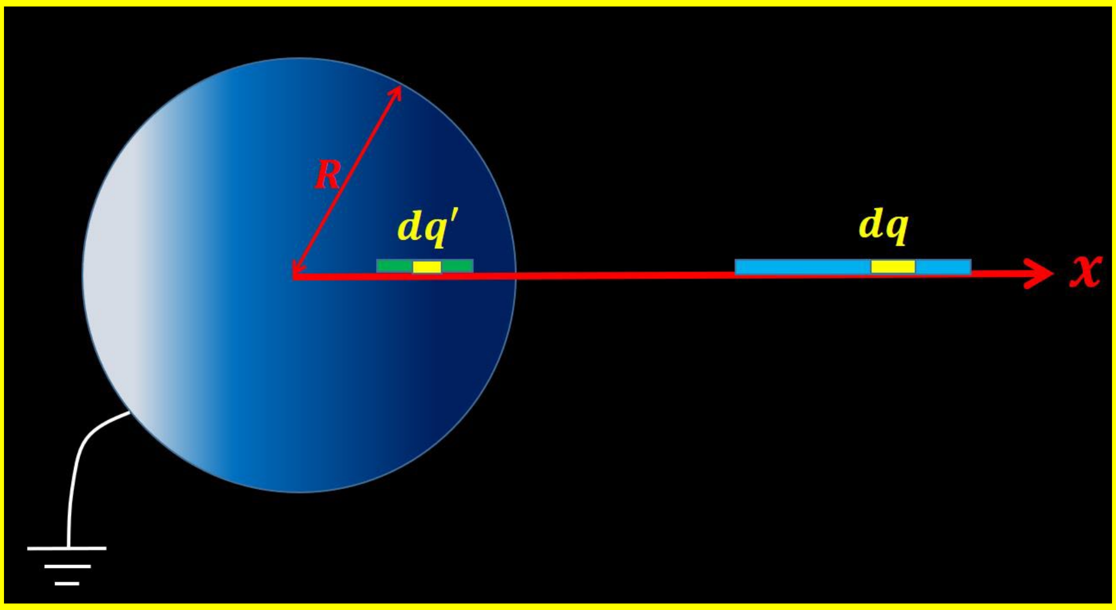


Fig. 6: A charged rod in front of a grounded conducting sphere

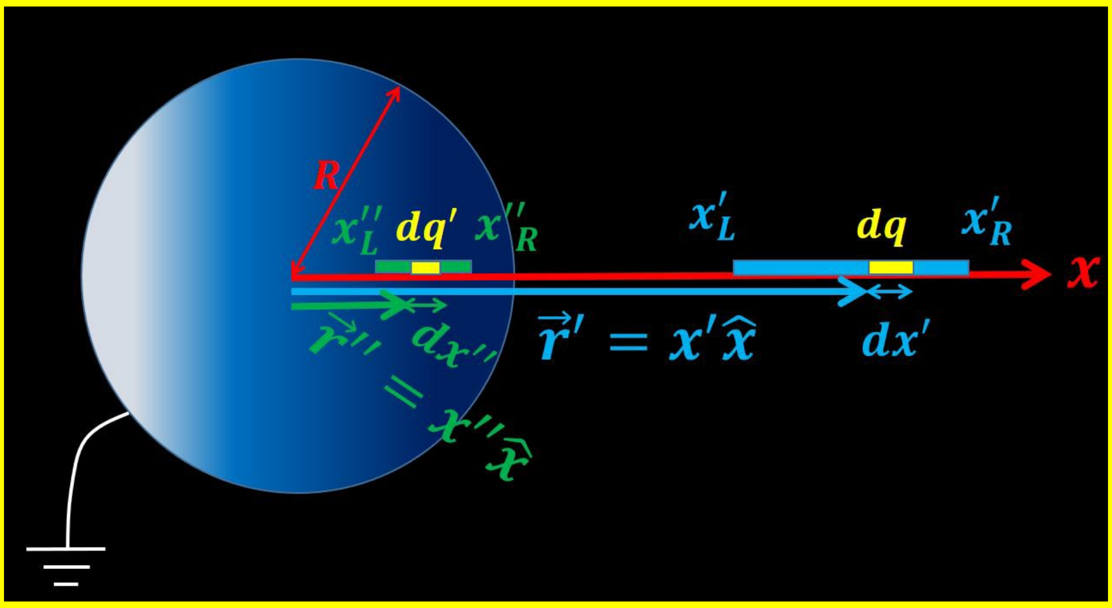


Fig. 7: Parametrization of the system of charged rod in front of a grounded conducting sphere

Then, we can extend Eqs. (2) and (3) to infinitesimal portions of the rod and write:

$$dq' = -\frac{R}{x'}dq \tag{7}$$

$$x'' = \frac{R^2}{x'} \tag{8}$$

These two equations give a few interesting results. First, we note that (7) implies that even though this rod is uniformly charged, its image (another rod inside the sphere) is nonuniformly charged!

The charge density can be worked out as:

$$\lambda'(x'')dx'' = -\frac{R}{x'}\lambda(x')dx' \tag{9}$$

Note that x' and x'' are both measured from the center of the sphere, therefore, dx' should have the same sign as dx'', hence:

$$\lambda'(x'') = -\frac{R}{x'}\lambda(x')\left|\frac{dx'}{dx''}\right| = -\frac{R}{\frac{R^2}{x''}}\lambda\left(\frac{R^2}{x''}\right)\frac{R^2}{x''^2} = -\frac{R}{x''}\ \lambda\left(\frac{R}{x''}\right) \tag{10}$$

(do you really see something like this in a calculus course!!!). For our uniformly charged rod of length L, density λ_0, Eq (10) gives:

$$\lambda'(x'') = -\frac{R}{x''}\lambda_0 = -\frac{R}{\frac{R^2}{x'}}\lambda_0 = -\frac{\lambda_0}{R}x' \tag{11}$$

Secondly, using Eq. (8) we can calculate the length of the image:

$$L' = x''_R - x''_L = \frac{R^2}{x'_L} - \frac{R^2}{x'_R} = \frac{R^2}{x'_L x'_R}(x'_R - x'_L) = \frac{R^2}{x'_L x'_R}L \tag{12}$$

which implies than not only does L' depend on L, it also depends on the location of the rod!

Thirdly, we can calculate the total image charge:

$$q' = \int_{x''_L}^{x''_R} \lambda'(x'')dx'' = -R\lambda_0 \int_{x''_L}^{x''_R} \frac{dx''}{x''} = -R\lambda_0\, ln\left(\frac{x''_R}{x''_l}\right) \tag{13}$$

Let's test this solution! How about pushing the rod to the point-charge limit and see if (13) recovers (2)?

We can test our formula to see if it recovers the point charge case. Using $L' = x''_R - x''_L$, and that in the point charge limit: $x'_R \to x'_L = x'$, $x''_R \to x''_L = x''$, we have:

$$q' = -R\lambda_0\, ln\left(\frac{x''_R}{x''_l}\right) = -R\lambda_0\, ln\left(\frac{x''_L + L'}{x''_L}\right) = -R\lambda_0\, ln\left(1 + \frac{L'}{x''_L}\right)$$

Then: (14)

$$\lim_{L' \to 0} q' = -R\lambda_0 \frac{L'}{x''} = -R\lambda_0 \frac{\frac{R^2}{x'^2}L}{\frac{R^2}{x'}} = -\frac{R\lambda_0 L}{x'} = -\frac{R}{x'}\, q \qquad (15)$$

With the confidence that Eq. (15) gives us, we are now ready to go further. We can calculate the potential at an arbitrary field point $\vec{r}$ as shown in Fig. 8.

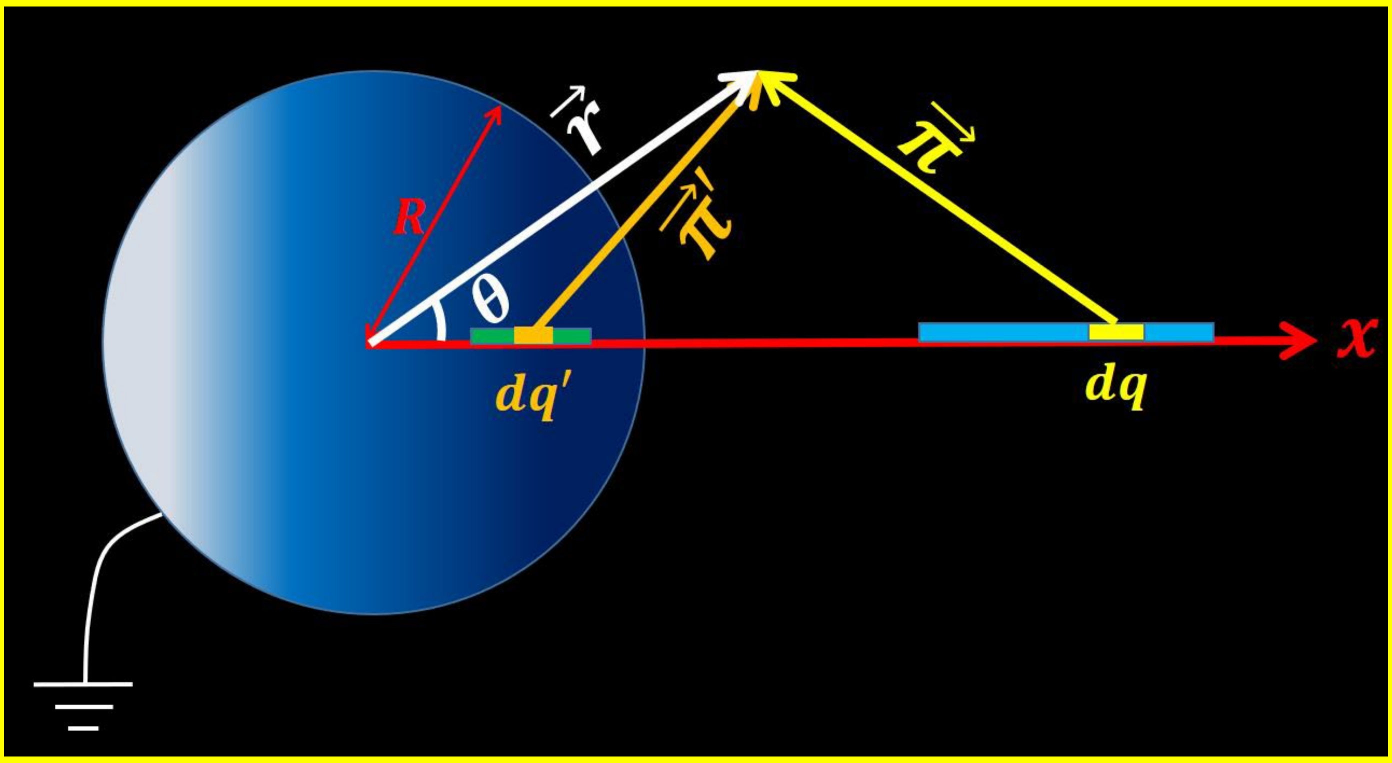


Fig. 8: Electric potential at point $\vec{r}$ produced by this system.

This potential is produced by the rod and its image.

$$V(r) = \frac{1}{4\pi\epsilon_0}\left[\int_{x''_L}^{x''_R} \frac{\lambda'}{|\vec{r} - \vec{r}_{q'}|}\, dx'' + \int_{x'_L}^{x'_R} \frac{\lambda_0}{|\vec{r} - \vec{r}_q|}\, dx'\right]$$

(16)

$$V(r) = \frac{1}{4\pi\epsilon_0}\left[\int_{x''_L}^{x''_R} \frac{\frac{-\lambda_0 R}{x''}}{\sqrt{r^2 + x''^2 - 2rx''cos\theta}}\,dx''\right.$$

$$\left.+\int_{x'_L}^{x'_R} \frac{\lambda_0}{\sqrt{r^2 + x'^2 - 2rx'cos\theta}}\,dx'\right] \qquad (17)$$

Using this we can then calculate the charge density on the surface of the sphere:

$$\sigma = -\epsilon_0 \frac{\partial V(r)}{\partial r}\bigg|_{r=R} \qquad (18)$$

and consequently the total surface charge on the conductor:

$$q' = \int_0^{\pi} \sigma\, 2\pi\, sin\theta\, d\theta$$

(19)

Let us check to see if (19) agrees with (13). Integrals (17) and (19) are rather lengthy, so we consider a semi-numeric approach. With: $R = 1$, $x'_L = 2$, $x'_R = 3$ and $\lambda_0 = 1$, Maple is used to compute the integrals (17) and (19) – the Maple code is given in the Appendix.

The charge density is a very complicated function of θ as displayed in the next page. The plot of charge density vs θ is also given in next page (Fig. 9).

The charge density is:

$$\begin{aligned}
sigma_num := \Bigg(&-\Big(\big((-2\cos(\theta)+4)\sqrt{-4\cos(\theta)+5}+\cos(\theta)^2-8\cos(\theta)+9\big)\left(\cos(\theta)-\frac{5}{4}\right)\left(\cos(\theta)-\frac{5}{3}\right)\big((-2\cos(\theta)+6)\sqrt{-6\cos(\theta)+10}+\cos(\theta)^2 \\
&-12\cos(\theta)+19\big)\ln\left(\sqrt{-6\cos(\theta)+10}-\cos(\theta)+3\right)\Big)+\Big(\big((-2\cos(\theta)+4)\sqrt{-4\cos(\theta)+5}+\cos(\theta)^2-8\cos(\theta)+9\big)\left(\cos(\theta)-\frac{5}{4}\right)\left(\cos(\theta)-\frac{5}{3}\right)\big((-2\cos(\theta) \\
&+6)\sqrt{-6\cos(\theta)+10}+\cos(\theta)^2-12\cos(\theta)+19\big)\ln\left(\sqrt{-4\cos(\theta)+5}-\cos(\theta)+2\right)\Big)+\Bigg(\left(\left(-\frac{39\cos(\theta)}{2}-\frac{31\cos(\theta)^3}{6}+\frac{55\cos(\theta)^2}{3}+5\right)\sqrt{-4\cos(\theta)+5}\right. \\
&\left.-\left(\cos(\theta)-\frac{5}{4}\right)\left(\cos(\theta)^4-4\cos(\theta)^3+\frac{52\cos(\theta)^2}{3}-\frac{92\cos(\theta)}{3}+11\right)\right)\sqrt{-6\cos(\theta)+10}+\Big(\big(\cos(\theta)^4+4\cos(\theta)^3-26\cos(\theta)^2+34\cos(\theta) \\
&-9\big)\sqrt{-4\cos(\theta)+5}-4\cos(\theta)^4+25\cos(\theta)^3-77\cos(\theta)^2+85\cos(\theta)-25\Big)\left(\cos(\theta)-\frac{5}{3}\right)\Bigg)\Bigg)\Bigg/\Bigg(4\Big(\big((-2\cos(\theta)+4)\sqrt{-4\cos(\theta)+5}+\cos(\theta)^2-8\cos(\theta) \\
&+9\big)\pi\left(\cos(\theta)-\frac{5}{4}\right)\left(\cos(\theta)-\frac{5}{3}\right)\big((-2\cos(\theta)+6)\sqrt{-6\cos(\theta)+10}+\cos(\theta)^2-12\cos(\theta)+19\big)\Big)\Bigg)
\end{aligned}$$

and its plot vs θ is given in Fig. 9 below.

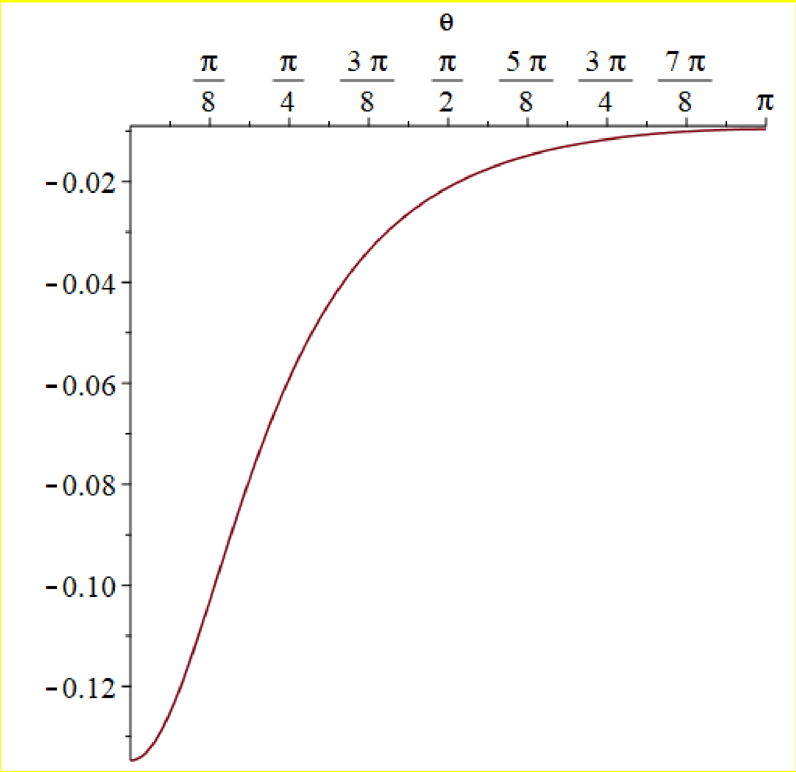


Fig. 9: Charge density on the surface of the sphere vs θ.

The same Maple code computes the total surface charge and finds:

$$q' = -ln\left(\frac{3}{2}\right) \qquad (20)$$

in complete agreement with Eq. (13)! 🙂

Now, let us compare the charge density produced by the rod with the charge density produced by a point charge. For numerical comparison, we take the total charge on the rod q equal to the total charge of the point charge placed at $x' = 2.0$ (left end of the rof), $x' = 2.5$ (middle of the rod), $x' = 3.0$ (right end of the rod) - see Fig. 10 (left). In Eq. (6) we substitute $q = 1, R = 1,\ a = 2.5$, and plot the density given by (6) vs θ as shown in Fig. 10(right).

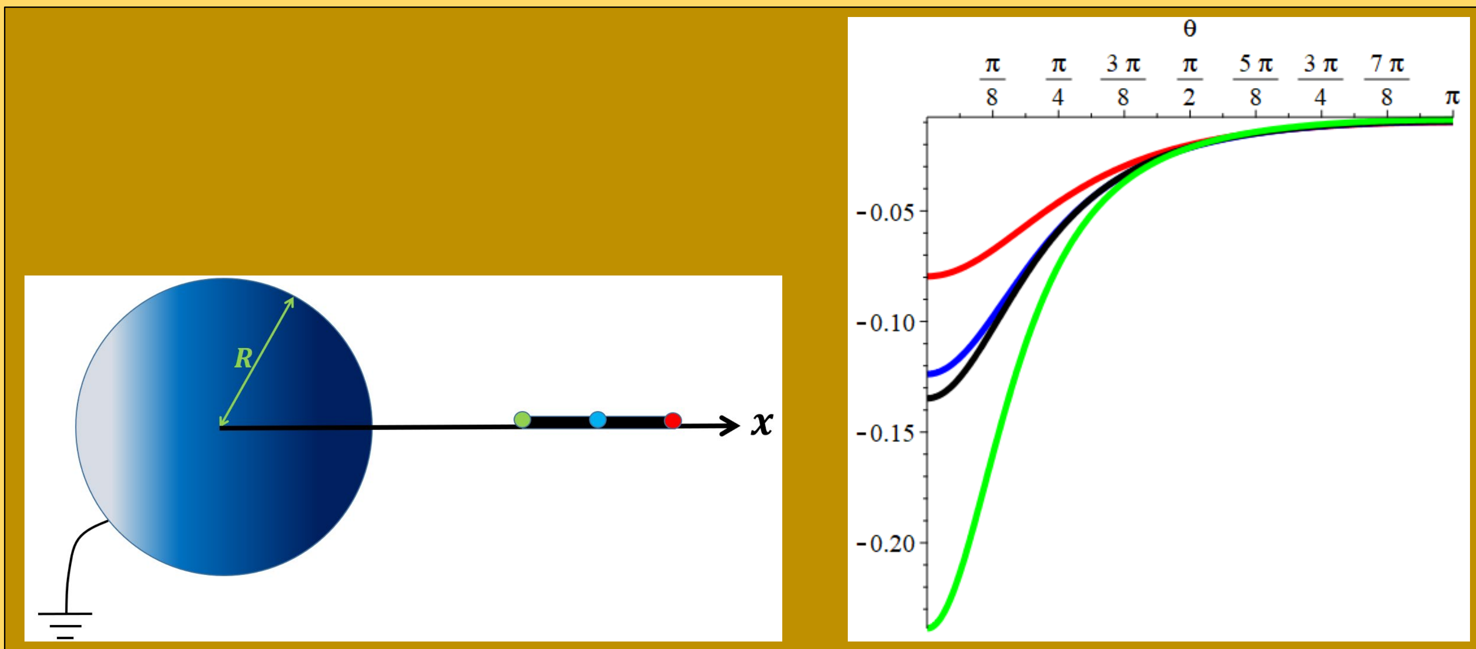


Fig. 10: Comparing the rod and point charge systems (left). The rod has the same total charge as each of the point charges. The right fig. compares the surface charge density on the sphere produced by the rod (black curve) with the same quantity produced by each of the point charges (the fig. colors on the right are produced by charges with the same color in the left fig.).

Fig. 10 raises a question: Is there any point that we can place the point charge so that the charge density produced by the rod becomes identical (or very close) to that produced by the point charge? We can calculate the location by noting that if the densities are to be the same, the total image charge should be the same too! This means, using Eqs. (2) and (20):

$$q' = -ln\left(\frac{3}{2}\right) = -\frac{Rq}{a} \qquad (21)$$

which, within our choice of units we find:

$$a = \frac{1}{ln\left(\frac{3}{2}\right)} = 2.466303462 \qquad (22)$$

With this input the charge density produced by this point charge is compared with the rod again in Fig. 11

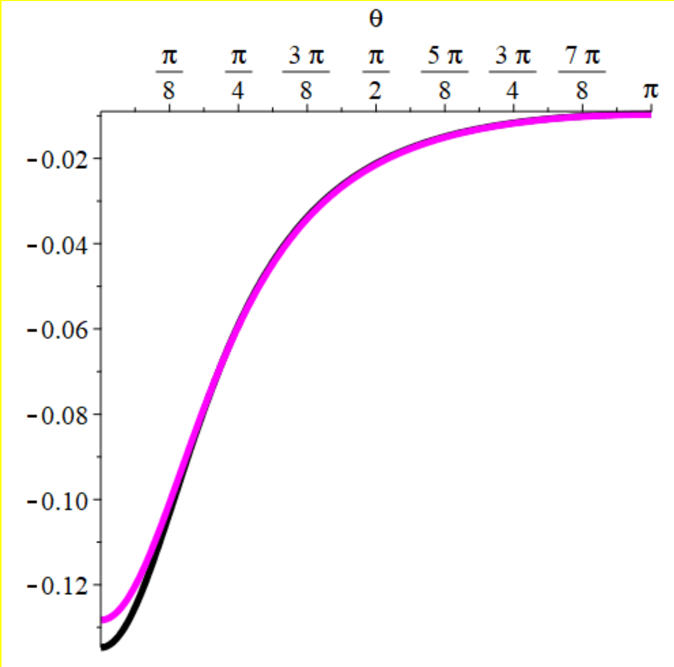


Fig. 11: Comparing the surface charge density produced by a charged rod (black) and produced by a point charge (magenta). The total image charge is the same in either case.

Note that in this case the total charge on the rod, and of the point charge are the same. Also the total image charges are the same. Fig. 11 shows some difference in density near $\theta = 0$, but this is compensated by a tiny difference between the densities for larger values of θ which cannot immediately seen in the figure. In other words, the black curve is lower than magenta near $\theta = 0$ but it is higher for larger values of θ, resulting in identical "total" image charges (if you look closely, you might be able to see this in the fig. as well).

What we considered here was the simplest case of an extended charged object and its fascinating calculus. In the next poster, we will work out the theory for the general case and then we will consider several nontrivial cases, such as, for example, a charged vertical rod, a charged slanted rod, a charged ring, a charged slanted ring, a charged plate, etc.

Finally, it is hard not to notice the beauty of the underlying calculus here, and even more so when we will consider the more general cases. This is why the best way of learning calculus is to study physics. No surprise here, of course, because calculus was born in physics, and physics will teach us the best calculus, forever!

Appendix: MAPLE Code

```
with(plots):
# Basic Inputs:
R := 1:
xp_L := 2:
xp_R := 3:
lambda0 := 1:
VL :=
1/(4*Pi*epsilon0)*lambda0*
int(
   1/sqrt(r^2 + xp^2 - 2*r*xp*cos(theta)),
   xp = xp_L .. xp_R):
xpp_L :=  R^2/xp_R:
xpp_R :=  R^2/xp_L:
VLp :=
-1/(4*Pi*epsilon0)*R*lambda0*
int(
   1/(xpp*sqrt(r^2 + xpp^2 - 2*r*xpp*cos(theta))),
   xpp = xpp_L .. xpp_R):
V_total := simplify(VL + VLp):
V_total_diff:= -epsilon0*simplify(diff(V_total, r)):
sigma_rod := simplify(subs(r=R, V_total_diff));
qp_rod := simplify(int(sigma_rod*2*Pi*sin(theta), theta = 0 .. Pi));
#  Plots:
P1 := plot(sigma_rod, theta = 0 .. Pi, color = black, thickness = 5):
sigma_point_25 := 1/(4*Pi)*(1 - 2.5^2)*(1 + 2.5^2 - 2*2.5*cos(theta))^(-3/2):
P2 := plot(sigma_point_25, theta = 0 .. Pi, color = blue, thickness = 5):
sigma_point_20 := 1/(4*Pi)*(1 - 2^2)*(1 + 2^2 - 2*2*cos(theta))^(-3/2):
P3 := plot(sigma_point_20, theta = 0 .. Pi, color = green, thickness = 5):
sigma_point_30 := 1/(4*Pi)*(1 - 3^2)*(1 + 3^2 - 2*3*cos(theta))^(-3/2):
P4 := plot(sigma_point_30, theta = 0 .. Pi, color = red, thickness = 5):
display({P1, P2, P3, P4});
a_exact := evalf(1/ln(3/2)):
sigma_point_exact := 1/(4*Pi)*(-a_exact^2 + 1)*(1 + a_exact^2 - 2*a_exact*cos(theta))^(-3/2):
P5 := plot(sigma_point_exact, theta = 0 .. Pi, color = magenta, thickness = 5):
display(P1, P5);
```