

Mechanics and Structure Preserving Discretizations of Shell Elasticity

Many phenomena in physical, engineering, and biological systems are modeled by elastic shells. Structure preserving algorithms proved to be extremely useful in a variety of applications, for example in classical field theory, plasma physics, or computer vision and graphics. However, similar discretization schemes for shell elasticity are yet to be designed.

The purpose of this workshop is to bring together experts in mathematics, theoretical mechanics, and numerical methods, to discuss and foster the development of structure-preserving algorithms applicable to elastic continua described by partial differential equations. In particular, we will focus on algorithms relevant to elastic shells.

A. Dziubek, M. Karow, C. Stoica, E. Rusjan

Wednesday, 8/28

9:30

Lecture
Exercises
Discussions

Shell Elasticity and Cosserat Theory

Michael Karow, TU-Berlin, Germany
Andrea Dziubek, SUNY Poly, Utica

10:50

Break

11:11

Lecture
Discussion

12:30

Light Lunch

1:30

Lecture
Exercises
Discussions

2:50

Break

3:10

Lecture
Exercises
Discussions

4:10

Dinner
at 5 pm

Elastic shells can be modeled as two-dimensional elastic bodies with a reference shape not necessarily flat and that can withstand both membrane (in plane) and bending (out-of-plane) effects.

In 1909, the Cosserat brothers [1], based on earlier work of Love [2] and Kirchhoff [3], were the first to derive the governing equations of elastic shells in terms of stress and couple stress tensors. In contrast to classical Continuum Theory –where the smallest constitutive element of media is the point-particle–, Cosserat theory, generalized by Naghdi [4], Eringen [5] and others, takes shell media to be a two-dimensional medium with microstructure that has additional rotational degrees of freedom, so called directors. Thus, translations and rotations have a similar status.

These equations were cast into intrinsic form by Koiter [6] where the (membrane and bending) deformation of the mid-surface of the shell is completely specified by the differences between the first and second fundamental forms in the deformed configuration and the undeformed configuration. The Koiter model can be seen as a special case of the (not intrinsic) model proposed by Reissner [7] and Mindlin [8] which takes transverse-shear stress into account. Cosserat and other shell models are discussed in detail by Naghdi [4].

While Ciarlet [9] and coworkers have shown that these shell models are well-posed, Finite Element implementations of shell models are often ill-conditioned, mostly because of singularities that appear as the shell thickness tends to zero.

Thursday, Friday, 8/29-30

Rudiments of Geometric Methods

Cristina Stoica, Wilfrid Laurier, CA

The formulations of Lagrangian and Hamiltonian mechanics in the coordinate-free language of differential geometry provide a unifying framework for many seemingly disparate physical systems, such as N -particle systems, rigid bodies, fluids and other continua, and electromagnetic and quantum systems. A famous example is that of the ideal flow: in 1966, V.I. Arnold observed that Euler ideal fluid motion may be identified with geodesic flow on the volume-preserving diffeomorphisms, with a metric determined by the fluid’s kinetic energy [10]. Moreover, ideal fluid motion may be understood as the infinite-dimensional counterpart of the free rigid body dynamics. The methods of geometric mechanics systematically developed this result from the Euler-Poincaré variational principle and generated the understanding of the associated Lie-Poisson Hamiltonian structure. Further, these theoretical developments are the foundations of later developed structure preserving and symplectic discretizations of ideal fluid flow.

Three-dimensional elastic continua are described in the geometric mechanics language in the textbook of Marsden and Hughes [11]. The Hamiltonian structure of shell dynamics was developed shortly after that by Simo et al. [12]. However, it was not until recently, and motivated by the search for structure preserving numerical methods, that there has been a renewed interest in the multisymplectic description of elasticity. Boyer et al. seems to be the first who applied the abstract variational calculus developed by Poincaré to Cosserat shell models [13].

Thursday lecture, discussions start at 1:30pm

Saturday, 8/31

Exterior Calculus and Structure-Preserving Discretizations

Michael Karow, TU-Berlin, Germany
Edmond Rusjan, SUNY Poly, Utica

Exterior calculus is a generalization of calculus to manifolds and allows the definition of differential operators in a coordinate invariant manner, which can then be discretized by intrinsic computation of quantities in the finite elements approximating the shell surface. For computational purposes, exterior calculus has been discretized as finite element exterior calculus (FEEC) [14] and discrete exterior calculus (DEC) [15, 16].

Research in this direction is currently very active and has been gaining increasingly attention from theoretical engineering communities [17, 18, 19, 20, 21]. Recent progress has been made on the discrete counterparts of essential operators in elasticity: the connection, the covariant derivative, and the stress-tensor (a co-vector-valued two-form) [22, 23, 24].

With their interconnection of differential geometry and continuum mechanics, shells are a natural place to study covariant discretization methods of metric dependent operators. However, the covariant discretization of shells is not yet formulated.

In shell models, the strain term is given by the tangential part of the covariant derivative and the bending term is given by normal part of the second covariant derivative of the displacement vector. In order to translate these in the language of exterior calculus/the language of differential forms we aim to clarify the covariant exterior derivative of a vector field (in contrast to the exterior derivative of a scalar field).

Lectures, exercises, discussions start at 10:50am

Sunday, 9/1

Workshops, Discussions all day

References

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Lectures, exercises, discussions start at 11:00am